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GAUSSIAN PROCESSES: NONLINEAR ANALYSIS AND STOCHASTIC CALCULUS.(U)
1977 S CAMBANIS, S T HUANG AFOSR-75-2796

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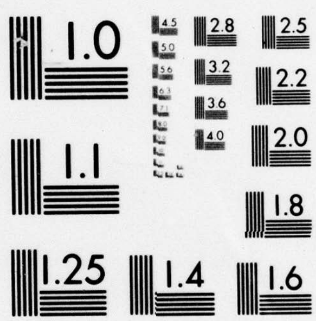


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# GAUSSIAN PROCESSES: NONLINEAR ANALYSIS AND STOCHASTIC CALCULUS*

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## 1. Introduction

For the Wiener process a large number of results on its nonlinear analysis have been developed (see Wiener (1958) and McKean (1973)), as well as a fairly complete and rich stochastic calculus (see for instance Friedman (1976)). Since the Wiener process is a Gaussian martingale, it is natural to investigate the extent to which these or similar results are true for (general) Gaussian processes and for (general) martingales.

For martingales the corresponding stochastic calculus is now well developed (see Kunita and Watanabe (1967) and Meyer (1976)). The nonlinear analysis and a stochastic calculus for Gaussian processes have been the subject of [3], [4] and [5]. This article is a survey of these references and its purpose is to make the main results and the basic ingredients of the approach easily accessible to the reader.

The basis of the approach is provided by the structure of the nonlinear space of a Gaussian process as developed by Kakutani (1961), Neveu (1968) and Kallianpur (1970); this is reviewed in Section 2. Sections 3 and 4 include results on the nonlinear analysis of Gaussian processes. Results currently available on the stochastic calculus of Gaussian processes are presented in Sections 5 to 8. The definition of the stochastic integral is in Section 6, its main properties in Section 5, a useful Riemann-like expression in Section 7, and the differential formula in Section 8. Stochastic differential equations with (general) Gaussian noise are currently under study.

The material in Sections 3, 4 (second half), 5 and 6 is from [3]; the first half of Section 4 is from [4]; and Sections 7 and 8 are from [5].

* This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-75-2796.

## 2. The Nonlinear Space of a Gaussian Process

The following notation and terminology will be used throughout. $X = (X_t, t \epsilon T)$ is a Gaussian process with mean zero and covariance function $R(t,s)$, defined on a probability space $(\Omega, \mathcal{B}, P)$, and $T$ is an interval on the real line (even though more general index sets could be used). $\mathcal{B}(X)$ is the $\sigma$-field generated by the random variables of $X$. The nonlinear space of $X$, $L_2(X) = L_2(\Omega, \mathcal{B}(X), P)$, consists of all $\mathcal{B}(X)$-measurable random variables with finite second moment which are called (nonlinear) $L_2$-functionals of $X$. The linear space of $X$, $H(X)$, is the closed subspace of $L_2(X)$ spanned by $X_t$, $t\epsilon T$, and its elements are called linear $L_2$-functionals of $X$. For each $p = 1,2,\ldots$, $P_p(X)$ is the set of all polynomials in the elements of $H(X)$ with degree $\leq p$, $P_0(X)$ is the set of constants, $\mathcal{Q}_p(X)$ is the set of all polynomials in $P_p(X)$ which are orthogonal to $P_{p-1}(X)$, $\mathcal{Q}_0(X) = P_0(X)$, and the closure $\overline{\mathcal{Q}}_p(X)$ of $\mathcal{Q}_p(X)$ in $L_2(X)$ is called the p-th homogeneous chaos. For $p \neq q$, $\overline{\mathcal{Q}}_p(X) \perp \overline{\mathcal{Q}}_q(X)$.

The structure of the nonlinear space of a Gaussian process is given by the following relationship, which can be found in [7] or [11], and which forms the basis of our analysis,

$$L_2(X) = \bigoplus_{p=0}^{\infty} \overline{\mathcal{Q}}_p(X) \cong \bigoplus_{p=0}^{\infty} H^{\tilde{\otimes} p}(X) .$$

Here $\otimes$ denotes tensor product and $\tilde{\otimes}$ symmetric tensor product. $\cong$ means "is isomorphic to" and the isomorphism $\Phi$ from $\oplus_{p=0}^{\infty} H^{\tilde{\otimes} p}(X)$ onto $L_2(X)$ maps each $H^{\tilde{\otimes} p}(X)$ onto $\overline{\mathcal{Q}}_p(X)$. If $\xi \epsilon H(X)$ then

$$\Phi\{\exp(\tilde{\otimes}\xi)\} = \exp(\xi - \tfrac{1}{2}E\xi^2) ,$$

where $\exp(\tilde{\otimes}\xi) = \sum_{p=0}^{\infty} (p!)^{-\frac{1}{2}} \xi^{\tilde{\otimes} p}$, and if $\xi_1,\ldots,\xi_k \epsilon H(X)$ are orthogonal then

$$\Phi(\xi_1^{\otimes p_1} \tilde{\otimes} \ldots \tilde{\otimes} \xi_k^{\tilde{\otimes} p_k}) = (p!)^{-\frac{1}{2}} \prod_{j=1}^{k} H_{p_j, E\xi_j^2}(\xi_j) ,$$

where $p = p_1 + \ldots + p_k$. $E$ denotes expectation, and the Hermite polynomial $H_{p,\sigma^2}$ with degree $p$ and parameter $\sigma^2$ is defined as follows: $\{H_{p,\sigma^2}(\xi), p=0,1,2,\ldots\}$ is obtained by applying the Gram-Schmidt procedure to orthogonalize the sequence of random variables $\{\xi^p, p=0,1,2,\ldots\}$ in $L_2(\xi)$, where $\xi$ is a Gaussian variable with mean zero and variance $\sigma^2$.

3. Multiple Wiener Integrals (MWI's)

In order to introduce the MWI's or order p=1,2,...

$$I_p(f_p) = \int_T \cdots \int_T f_p(t_1,\ldots,t_p)dX_{t_1}\ldots dX_{t_p} = \int_{T^p} f_p(\underline{t})dX_{\underline{t}}^p$$

we first have to define the space of (deterministic) integrands $\Lambda_2(\otimes^p R)$.

For p=1, $\Lambda_2(R)$ is the completion of the space of all step functions on T, $f(t) = \sum_{n=1}^N f_n 1_{(a_n,b_n]}(t)$, with respect to the inner product

$$\langle f,g\rangle = \iint_{TT} f(t)g(s)d^2R(t,s) = \sum_{n=1}^N \sum_{m=1}^M f_n g_m\{R(b_n,d_m) + R(a_n,c_m) - R(a_n,d_m) - R(b_n,c_m)\}$$

(where $g(t) = \sum_{m=1}^M g_m 1_{(c_m,d_m]}(t)$). Thus $\Lambda_2(R)$ is a Hilbert space of "functions" on T; it contains those functions f for which the Riemann integral $\iint f(t)f(s)d^2R(t,s)$ exists, and when R is of bounded variation it contains all bounded measurable functions. When $R(t,s) = \min(t,s)$, $\Lambda_2(R) = L_2(T,dt)$. The MWI of order 1, $I_1:\Lambda_2(R) \to H(X)$, is the isometry into H(X) defined by

$$I_1\left(\sum_{n=1}^N f_n 1_{(a_n,b_n]}\right) = \sum_{n=1}^N f_n(X_{b_n} - X_{a_n}) .$$

$I_1$ becomes onto H(X), hence an isomorphism, if $X_{t_0} = 0$ a.s. for some $t_0 \in T$, as we now assume (otherwise replace H(X) by H(ΔX), the linear space of the increments ΔX of X).

$\Lambda_2(\otimes^p R)$ is defined similarly, by starting with step functions on $T^p$, and is isomorphic to $\otimes^p \Lambda_2(R)$. $\Lambda_2(\tilde{\otimes}^p R)$ is the subspace of all symmetric "functions" in $\Lambda_2(\otimes^p R)$ (a concept defined again starting with step functions) and $\Lambda_2(\tilde{\otimes}^p R) \cong \tilde{\otimes}^p \Lambda_2(R)$. Since $\Lambda_2(R) \cong H(X)$ under $I_1$, $\Lambda_2(\tilde{\otimes}^p R) \cong \tilde{\otimes}^p \Lambda_2(R)$ is isomorphic to $H^{\tilde{\otimes} p}(X)$ and we denote this isomorphism by $I^{\tilde{\otimes} p}$. Then the MWI of order p, $I_p: \Lambda_2(\tilde{\otimes}^p R) \to \overline{\mathcal{Q}}_p(X)$, is defined by

$$I_p = (p!)^{\frac{1}{2}}\Phi \cdot I^{\tilde{\otimes} p}$$

and is extended to $\Lambda_2(\otimes^p R)$ by $I_p(f) = I_p(\tilde{f})$ where $\tilde{f}$ is the symmetric tensor of $f \in \Lambda_2(\otimes^p R)$.

Thus each MWI $I_p$ is a bounded linear operator from $\Lambda_2(\otimes^p R)$ onto $\overline{\mathcal{Q}}_p(X)$, with the following properties:

$$E\{I_p(f)I_p(g)\} = p! \ \langle\tilde{f},\tilde{g}\rangle_{\Lambda_2(\otimes^p R)} ,$$

$$\mathcal{E}\{I_p(f)I_q(g)\} = 0 \quad \text{if } p \neq q$$

$$I_p(\phi_1^{\tilde{\otimes} p_1} \tilde{\otimes} \ldots \tilde{\otimes} \phi_k^{\tilde{\otimes} p_k}) = \prod_{j=1}^{k} H_{p_j, ||\phi_j||^2}(\int \phi_j dX) \ ,$$

where $\{\phi_1, \ldots, \phi_k\}$ is an orthogonal set in $\Lambda_2(R)$ and $p_1 + \ldots + p_k = p$. Also every $L_2$-functional $\theta$ of X, $\theta \epsilon L_2(X)$, has an orthogonal development

$$\theta - \mathcal{E}\theta = \sum_{p=1}^{\infty} I_p(f_p) = \sum_{p=1}^{\infty} \int_{T^p} f_p(\underline{t}) dX_{\underline{t}}^p$$

for some $f_p \epsilon \Lambda_2(\otimes^p R)$, and if $\theta - \mathcal{E}\theta = \sum_{p=1}^{\infty} I_p(f_p) = \sum_{p=1}^{\infty} I_p(g_p)$ then $\tilde{f}_p = \tilde{g}_p$, $p \geq 1$.

MWI's of the following type can also be defined

$$J_p(f_p) = \int_T \ldots \int_T f_p(t_1, \ldots, t_p) X_{t_1} \ldots X_{t_p} dt_1 \ldots dt_p = \int_{T^p} f_p(\underline{t}) X_{\underline{t}}^p \, d\underline{t}$$

for $f_p \epsilon \lambda_2(\otimes^p R)$ with similar properties when X is mean square continuous. Finally the MWI's of both types can be evaluated from the sample paths of X.

## 4. Nonlinear Systems with Gaussian Inputs

Consider a nonlinear system with input the mean square continuous Gaussian process $X = \{X_t, t \epsilon T\}$ and output the second order process $Y = \{Y_t, t \epsilon T\}$, i.e. the only assumption on the system is that $Y_t \epsilon L_2(X)$, $t \epsilon T$. Then, by Section 3, the output Y can be represented by

$$Y_t = \mathcal{E}Y_t + \sum_{p=1}^{\infty} \int_T \ldots \int_T f_p(t; t_1, \ldots, t_p) X_{t_1} \ldots X_{t_p} dt_1 \ldots dt_p$$

where $f_p(t;\cdot) \epsilon \lambda_2(\tilde{\otimes}^p R)$. The action of the system to the input X is thus represented by the sequence of kernels $\{f_p\}_{p=1}^{\infty}$ which depends on the input X (distinct input Gaussian processes will in general produce distinct sequences of kernels). These kernels can be determined from knowledge of the joint statistics of the input and output processes. Moreover, for almost every sample function of X as its input, theoutput of the nonlinear system has a Volterra representation (i.e. a series representation like above with the MWI's replaced by Lebesgue integrals) whose kernels can be found from the kernels $\{f_p\}$; i.e. assuming only that $\mathcal{E}Y_t^2 < \infty$, $t \epsilon T$, we have the remarkable result that for a small class of deterministic inputs (almost all sample functions of X) the nonlinear system has a Volterra input-output representation - a result obtained by Fréchet (1910) for large classes of inputs, like $C(T)$ or $L_2(T)$, when the system is continuous (i.e., the output at each fixed t, is a continuous functional on $C(T)$ or $L_2(T)$). Finally if the nonlinear system has a Volterra input-output representation with kernels $\{K_p\}$ when acting on deterministic inputs in $L_2(T)$, the relationship between the two sets of kernels $\{f_p\}$ and $\{K_p\}$ can be established. For the details see [4].

When the input Gaussian process X has stationary increments with say $X_0 = 0$ a.s., a more convenient representation of the system output is

$$Y_t = EY_t + \sum_{p=1}^{\infty} \int_{-\infty} \cdots \int_{-\infty} f_p(t;t_1,\ldots,t_p)dX_{t_1}\cdots dX_{t_p}$$

where $f_p(t;\cdot)\epsilon\Lambda_2(\otimes^p R)$. When the system is time invariant, in the sense that

$$f_p(t;t_1,\ldots,t_p) = g_p(t_1-t,\ldots,t_p-t)$$

then Y is strictly stationary and is called X-presentable. A natural question is how large is the class of X-presentable processes, or the class of processes which can be approximated by X-presentable processes. By introducing a Fourier transform in the spaces $\Lambda_2(\otimes^p R)$, results similar to those valid when X is the Wiener process can be proved:

(i) If X has absolutely continuous spectral distribution, then every X-presentable process is strongly mixing.

(ii) (the analogue of the Wiener-Nisio theorem). If X is sample continuous, ergodic, and satisfies an additional weak condition (valid when X has rational spectral density), then every measurable, ergodic, strictly stationary process is the limit in law of a sequence of X-presentable processes.

## 5. The Stochastic Integral and its Properties

The appropriate space of stochastic integrands f for the stochastic integral

$$I(f) = \int_T f(t)dX_t$$

is a generalization of the space $\Lambda_2(R)$ denoted by $\Lambda_{2;L_2(X)}(R)$. Like $\Lambda_2(R)$, $\Lambda_{2;L_2(X)}(R)$ is the completion of the space of all $L_2(X)$-valued step functions on T, $f(t) = \sum_{n=1}^N f_n 1_{(a_n,b_n]}(t)$, $f_n\epsilon L_2(X)$, with respect to the inner product

$$\langle f,g\rangle = \iint_{TT} E\{f(t)g(s)\}d^2R(t,s) = \sum_{n=1}^{N}\sum_{m=1}^{M} E\{f_n g_m\}\{R(b_n,d_m)+R(a_n,c_m)-R(a_n,d_m)-R(b_n,c_m)\}$$

(where $g(t) = \sum_{m=1}^{M} g_m 1_{(c_m,d_m]}(t)$, $g_m \epsilon L_2(X)$). Thus $\Lambda_{2;L_2(X)}(R)$ is a Hilbert space of "second order processes" on T and its properties are analogues of those of $\Lambda_2(R)$. In particular, when R is of bounded variation $\Lambda_{2;L_2(X)}(R)$ contains all measurable second order processes f(t) with $Ef^2(t)$ bounded, and if $R(t,s) = \min(t,s)$ then $\Lambda_{2;L_2(X)}(R) = L_{2;L_2(X)}(T,dt)$, the Hilbert space of all Lebesgue square integrable $L_2(X)$-valued functions on T. Since $L_2(X) = \oplus_{p=0}^{\infty} \mathcal{Q}_p(X)$ we have

$$\Lambda_{2;L_2(X)}(R) = \bigoplus_{p=0}^{\infty} \Lambda_{2;\mathcal{Q}_p(X)}(R) .$$

The stochastic integral

$$I: \quad \Lambda_{2;L_2(X)}(R) \to L_2^0(X) = L_2(X) \ominus \mathcal{Q}_0$$

is then an unbounded, densely defined, closed linear onto map. Its detailed definition is given in the next section. Here we summarize its basic properties. Each $\Lambda_{2;\mathcal{Q}_p(X)}(R)$ belongs to the domain $\mathcal{D}(I)$ of the stochastic integral which, when restricted to $\Lambda_{2;\mathcal{Q}_p(X)}(R)$, is a bounded linear operator onto $\mathcal{Q}_{p+1}(X)$ with norm $(p+1)^{\frac{1}{2}}$. If $f \epsilon \Lambda_{2;L_2(X)}(R)$ and $f = \sum_{p=0}^{\infty} f_p$, $f_p \epsilon \Lambda_{2;\mathcal{Q}_p(X)}(R)$, then $f \epsilon \mathcal{D}(I)$ if and only if $\sum_{p=0}^{\infty} E[I(f_p)]^2 < \infty$, in which case $I(f) = \sum_{p=0}^{\infty} I(f_p)$.

Since $I$ is onto $L_2^0(X)$, every $L_2$-functional $\theta$ of X, $\theta \epsilon L_2(X)$, has a stochastic integral representation

$$\theta = E\theta + \int_T f(t) dX_t$$

for some $f \epsilon \mathcal{D}(I)$. In fact f may be taken to be adapted to X, i.e. $f \epsilon \mathcal{D}(I) \cap \Lambda^{ad}_{2;L_2(X)}(R)$ where $\Lambda^{ad}_{2;L_2(X)}(R)$ is the closed subspace of $\Lambda_{2;L_2(X)}(R)$ generated by the simple functions adapted to X (i.e. $f(t) = \sum_{n=1}^{N} f_n 1_{(a_n,b_b]}(t)$ where each $f_n$ is $\sigma(X_t, t \le a_n)$-measurable). Thus the stochastic integral is defined for general (not necessarily adapted) integrands, and when X is the Wiener process it extends the Itô integral and it agrees with Skorokhod's (1975) generalization of the Itô integral to not necessarily adapted integrands.

A step function $f(t) = \sum_{n=1}^{N} f_n 1_{(a_n,b_n]}(t)$, $f_n \epsilon L_2(X)$, is called future increments independent (fii) if each $f_n$ is independent of the increments of X after $a_n$, and the closed subspace of $\Lambda_{2;L_2(X)}(R)$ generated by the fii step functions is denoted by $\Lambda^{fii}_{2;L_2(X)}(R)$. $\Lambda^{fii}_{2;L_2(X)}(R)$ belongs to the domain of the stochastic integral and when the stochastic integral is restricted to it it becomes norm preserving, like the Itô integral (its range $I(\Lambda^{fii}_{2;L_2(X)}(R))$ is not yet characterized).

As an indication of the calculation of the stochastic integral let us consider the simplest possible integrand $f(t) = \theta\phi(t)$, $\theta\epsilon L_2(X)$, $\phi\epsilon\Lambda_2(R)$. If $\theta$ and $\int_T\phi(t)dX_t$ are independent, then

$$\int_T \theta\phi(t)dX_t = \theta \int_T \phi(t)dX_t \ .$$

If $\theta\epsilon H(X)$, then

$$\int_T \theta\phi(t)dX_t = \theta \int_T \phi(t)dX_t - E(\theta \int_T \phi(t)dX_t) \ .$$

More important, when R is continuous and of bounded variation on $[a,b]\times[a,b]$ we have

$$(5.1) \qquad \int_a^b H_{p,\sigma_t^2}(X_t)dX_t = \frac{1}{p+1} \{H_{p+1,\sigma_b^2}(X_b) - H_{p+1,\sigma_a^2}(X_a)\} \ , \quad p \geq 0 \ ,$$

$$\int_a^b \exp(X_t-\tfrac{1}{2}\sigma_t^2)dX_t = \exp(X_b-\tfrac{1}{2}\sigma_b^2) - \exp(X_a-\tfrac{1}{2}\sigma_a^2)$$

where $\sigma_t^2 = EX_t^2 = R(t,t)$. Thus the Hermite polynomials $H_{p,\sigma_t^2}(X_t)$ play the role of customarary powers, $X_t^p$, and $\exp(X_t-\frac{1}{2}\sigma_t^2)$ the role of the customary exponential, $\exp(X_t)$, in this stochastic calculus.

The stochastic integral $\int f(t)dX_t$ for a Gaussian process X with mean function $m_t$ of bounded variation may be defined as $\int f(t)dm_t + \int f(t)d(X_t-m_t)$ when both integers exist. Here, the first integral is either the sample path or the mean square integral and the second is the stochastic integral for zero mean Gaussian process.

The relationship between this stochastic integral and some previously known integrals (e.g. when X is a Gaussian semi-martingale or when X has paths of bounded variation) is given in Sections 7 and 8.

## 6. The Definition of the Stochastic Integral

Here we present the definition of the stochastic integral $I$ and its domain $\mathcal{D}(I)$. $I$ is the composition of the maps

$$I = \Phi \cdot \Psi \cdot \Phi_0^{-1} \cdot I_{\otimes}$$

shown in the following diagram and defined below. It should be noted that $\Phi$ is the isomorphism between the nonlinear space of X and the symmetric tensor products of its linear space (and $\Phi_0$ is an isomorphism closely related to it) and if these two spaces were identified, as is usually done, $\Phi$ and $\Phi_0$ would not appear in the expression of $I$. The important ingredients of the stochastic integral $I$ are the tensor product integral $I_\otimes$ and the map $\Psi$ which "symmetrizes" in an appropriate way the tensor product of the integral $I_\otimes$.

$$\begin{array}{lclclcl}
\Lambda_{2;L_2(X)}(R) & \overset{I_\otimes}{\cong} & L_2(X)\otimes H(X) & \overset{\Phi_0^{-1}}{\cong} & \{\overset{\infty}{\underset{p=0}{\oplus}} H^{\tilde{\otimes} p}(X)\}\otimes H(X) & & \\
\mathcal{D}(I) & \overset{I_\otimes}{\cong} & H & \overset{\Phi_0^{-1}}{\cong} & \mathcal{D}(\Psi) & \xrightarrow[\text{onto}]{\Psi} & \overset{\infty}{\underset{p=1}{\oplus}} H^{\tilde{\otimes} p}(X) \overset{\Phi}{\cong} L_2^0(X) \\
\Lambda_{2;\bar{\mathcal{Q}}_p(X)}(R) & \overset{I_\otimes}{\cong} & \bar{\mathcal{Q}}_p(X)\otimes H(X) & \overset{\Phi_0^{-1}}{\cong} & H^{\tilde{\otimes} p}(X)\otimes H(X) & \xrightarrow[\text{onto}]{\Psi=\Psi_p} & H^{\tilde{\otimes} p+1}(X) \overset{\Phi}{\cong} \bar{\mathcal{Q}}_{p+1}(X) \ .
\end{array}$$

The tensor product integral $I_\otimes(f) = \int_T f(t)\otimes dX_t$ is the isomorphism from $\Lambda_{2;L_2(X)}(R)$ onto $L_2(X)\otimes H(X)$ defined for simple functions in $\Lambda_{2;L_2(X)}(R)$ by

$$I_\otimes\left(\sum_{n=1}^{N} f_n 1_{(a_n,b_n]}(t)\right) = \sum_{n=1}^{N} f_n \otimes (X_{b_n} - X_{a_n}) \ .$$

The isomorphism from $\oplus_{p=0}^{\infty} H^{\tilde{\otimes} p}(X)$ onto $L_2(X)$ has been denoted by $\Phi$, and $\Phi_0$ is the corresponding isomorphism from $\{\oplus_{p=0}^{\infty} H^{\tilde{\otimes} p}(X)\} \otimes H(X)$ onto $L_2(X) \otimes H(X)$. $\Psi_0$ is the natural isomorphism from $R\otimes H(X)$ onto $H(X)$, $\Psi_0(a\otimes\xi) = a\xi$, and for each $p=1,2,\ldots$, $\Psi_p$ is a bounded linear map from $H^{\tilde{\otimes} p}(X) \otimes H(X)$ onto $H^{\tilde{\otimes} p+1}(X)$ with norm $(p+1)^{1/2}$ defined as follows: if $\{\xi_\gamma, \gamma\in\Gamma\}$ is a complete orthonormal set (CONS) in $H(X)$ then

$$\Psi_p\{(\xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \ldots \tilde{\otimes} \xi_{\gamma_k}^{\tilde{\otimes} p_k}) \otimes \xi_\gamma\} = (p+1)^{1/2}\, \xi_{\gamma_1}^{\tilde{\otimes} p_1} \tilde{\otimes} \ldots \tilde{\otimes} \xi_{\gamma_k}^{\tilde{\otimes} p_k} \tilde{\otimes} \xi_\gamma \ , \ p=p_1+\ldots+p_k \ .$$

$\Psi = \oplus_{p=0}^{\infty} \Psi_p$ is the map from $\{\oplus_{p=0}^{\infty} H^{\tilde{\otimes} p}(X)\} \otimes H(X)$ onto $\oplus_{p=1}^{\infty} H^{\tilde{\otimes} p}(X)$ whose restriction to each $H^{\tilde{\otimes} p}(X) \otimes H(X)$ is $\Psi_p$. Since $||\Psi_p|| = (p+1)^{1/2}$ is unbounded in p, $\Psi$ is an unbounded, densely defined linear map with domain $\mathcal{D}(\Psi)$, the set of all $\phi\in\{\oplus_{p=0}^{\infty} H^{\tilde{\otimes} p}(X)\} \otimes H(X)$ such that $\sum_{p=0}^{\infty} ||\Psi_p(\phi_p)||^2 < \infty$ where $\phi = \sum_{p=0}^{\infty}\phi_p$, $\phi_p$ being the projection of $\phi$ onto $H^{\tilde{\otimes} p}(X) \otimes H(X)$. Then $H = \Phi_0\{\mathcal{D}(\Psi)\}$ and the domain of the stochastic integral is $\mathcal{D}(I) = I_\otimes^{-1}(H)$.

## 7. A Useful Expression for the Stochastic Integral

Here we express the stochastic integral of a certain class of integrand processes in terms of Riemann sums involving a tensor product denoted by $\odot$. This tensor product $\odot$ turns out to be the "natural" product of elements of $L_2(X)$, in view of its tensor product structure $\oplus_{p=0}^{\infty} H^{\tilde{\otimes} p}(X) \overset{\Phi}{\cong} L_2(X)$. Just as the usual product of elements of $L_2(X)$ may not be in $L_2(X)$, the $\odot$ product is not defined for all elements of $L_2(X)$. When $\xi_1, \ldots, \xi_k \epsilon H(X)$, $p_1, \ldots, p_k \geq 0$, and $p_1 + \ldots + p_k = p$, we define

$$H_{p_1, E\xi_1^2}(\xi_1) \odot \ldots \odot H_{p_k, E\xi_k^2}(\xi_k) = (p!)^{\frac{1}{2}} \Phi(\xi_1^{\tilde{\otimes} p_1} \tilde{\otimes} \ldots \tilde{\otimes} \xi_k^{\tilde{\otimes} p_k}) .$$

For general elements $\theta_1, \ldots, \theta_k$ of $L_2(X)$ the tensor product $\theta_1 \odot \ldots \odot \theta_k$ is defined through their Cameron-Martin expansions provided the resulting series converges.

It is easy to see that as a binary operation the tensor product $\odot$ is commutative, associative and bilinear. Therefore, the algebraic manipulation of $L_2$-functionals under the usual product still holds true under the tensor product $\odot$. The significance of the tensor product $\odot$ is that most algebraic formulae will have new analytic meaning in terms of Gaussian r.v.'s.

The explicit relationship between the tensor product $\odot$ and the usual product of certain $L_2$-functionals of X can be derived. Here are two examples. If $\theta_1, \theta_2 \epsilon L_2(X)$ are independent then

$$\theta_1 \odot \theta_2 = \theta_1 \theta_2 \epsilon L_2(X) .$$

If $F(x)$ and $G(x)$ are infinitely differentiable functions on R and if there exist constants c, d > 0 such that $|F^{(p)}(0)| + |G^{(p)}(0)| \leq cd^p$ for all $p \geq 0$, then for $\xi, \eta \epsilon H(X)$

(7.1)
$$F(\xi)G(\eta) = \sum_{p=0}^{\infty} \frac{(E\xi\eta)^p}{p!} F^{(p)}(\xi) \odot G^{(p)}(\eta) ,$$
$$F(\xi) \odot G(\eta) = \sum_{p=0}^{\infty} \frac{(-E\xi\eta)^p}{p!} F^{(p)}(\xi) G^{(p)}(\eta) .$$

Now consider the Riemann-Stieltjes tensor product integral $\int f(t) \odot dX_t$ for $f: T \to L_2(X)$. This integral can be defined as the mean square limit (if it exists and is unique) of the corresponding Riemann sums $\sum f(\bar{t}_i) \odot (X_{t_{i+1}} - X_{t_i})$ $(t_i \leq \bar{t}_i \leq t_{i+1})$. When dealing with this integral, we shall always assume that $T = [a,b]$, $f(t)$ is mean square continuous and $R(t,s)$ is of bounded variation on $[a,b] \times [a,b]$. When the integral $\int f(t) \odot dX_t$ exists then f belongs to the domain of the stochastic integral and the two integrals are equal:

(7.2) $$\int f(t)dX_t = \int f(t)\circledcirc dX_t \ .$$

If $f(t)\epsilon\overline{P}_p(X)$ for all $t\epsilon T$, then $f(t)\circledcirc dX_t$ exists. Also if $F(t,x)$ is a function on $[a,b]\times R$ continuous in t and infinitely differentiable in x, and if

(7.3) there exist $c,d > 0$ such that $\sup\limits_{a\le t\le b}\left|\frac{\partial^p}{\partial x^p} F(t,0)\right| < cd^p$ for all $p \ge 0$,

then $\int f(t)\circledcirc dX_t$ exists.

The equality between this Riemann-Stieltjes integral and the stochastic integral, (7.2), provides a way of evaluating the stochastic integral, and following are some examples.

1. A routine computation shows that

$$\int_a^b X_t^{\circledcirc p} \circledcirc dX_t = \frac{1}{p+1}(X_b^{\circledcirc p+1} - X_a^{\circledcirc p+1})$$

which is equivalent to (5.1).

2. Let $X \equiv W$ be the Wiener process and let $F(x)$ be a function satisfying (7.3). Given $u \ge 0$, $a \le c \le c+u \le b$, we have the following expression for the stochastic integral of the anticipating functional $F(W_{t+u})$:

$$\begin{aligned}\int_a^b F(W_{t+u})dW_t &= \lim \sum F(W_{\bar{t}_i+u}) \circledcirc (W_{t_{i+1}}-W_{t_i}) \\ &= \lim \sum \{F(W_{\bar{t}_i+u})(W_{t_{i+1}}-W_{t_i}) - F'(W_{\bar{t}_i+u})(t_{i+1}-t_i)\} \\ &= R\int_a^c F(W_{t+u})dW_t + \int_a^c F'(W_t)dt\end{aligned}$$

where $R\int_a^c F(W_{t+u})dW_t$ is the usual Riemann-Stieltjes integral in the mean square sense.

3. $$\begin{aligned}\int_a^b X_t\circledcirc dX_t &= \lim \sum X_{t_i} \circledcirc(X_{t_{i+1}}-X_{t_i}) \\ &= \lim \sum X_{t_{i+1}} \circledcirc(X_{t_{i+1}}-X_{t_i})\end{aligned}$$

implies that

$$\lim\{\sum(X_{t_{i+1}}-X_{t_i})^2 - \sum E(X_{t_{i+1}}-X_{t_i})^2\} = 0 \ .$$

Consequently, the (mean square) quadratic variation $V_a^b$ of X on [a,b] along any sequence of partitions whose mesh goes to zero exists and is given by

$$V_a^b = R(D_a^b)$$

where $D_a^b$ is the diagonal of $[a,b]\times[a,b]$ and R represents the signed measure corresponding to the covariance R(t,s). Furthermore, if the uniform limit of $\{R(t+u,t)-R(t,t)\}u^{-1}$, exists as $u\downarrow 0$ on $a\le t\le b$ and if it is denoted by $R_1(t,t)$, then

$$V_a^b = (\sigma_b^2-\sigma_a^2) - 2\int R_1(t,t)dt \ . \tag{7.4}$$

4. Suppose X is a solution of the stochastic differential equation

$$dX_t = a(t,X_t)dt + b(t)dW_t$$

where W is the Wiener process and a(t,x) and b(t) are smooth scalar functions, and that F(t,x) satisfies (7.3). Then the semi-martingale stochastic integral is

$$M\int F(t,X_t)dX_t = \int F(t,X_t)a(t,X_t)dt + \int F(t,X_t)b(t)dW_t$$

and the stochastic integral becomes

$$\int F(t,X_t)\circ dX_t = \int [F(t,X_t)\circ a(t,X_t)]dt + \int [F(t,X_t)\circ b(t)]dW_t \ .$$

It then follows that

$$\int F(t,X_t)\circ dX_t = M\int F(t,X_t)dX_t + \sum_{p=1}^{\infty} \frac{(-1)^p}{p!} \int \sigma_t^{2p} F_p(t,X_t)a_p(t,X_t)dt$$

where $F_p(t,x) = \partial^p F(t,x)/\partial x^p$, $a_p(t,x) = \partial^p a(t,x)/\partial x^p$. When $F(t,x) = a(t)x$ then X is a Guassian semi-martingale and

$$\int F(t,X_t)dX_t = M\int F(t,X_t)dX_t - \int \sigma_t^2 F_x(t,X_t)a(t)dt \ .$$

When $F(t,x) = a(t)$ then our stochastic integral and the semi-martingale integral are equal.

## 8. The Differential Formula

Suppose $T = [a,b]$. Suppose R(t,s) is continuous, of bounded variation and $\sigma_t^2 = R(t,t)$ is absolutely continuous. Suppose the uniform limit $\{R(t+u,s) - R(t,s)\}u^{-1}$, as $u\downarrow 0$, exists on $[a,b]\times[a,b]$ and is denoted by $R_1(t,s)$. Finally, suppose F(t,x) is continuously differentiable with respect to t and satisfies condition (7.3). Then we have the following differential formula

(8.1) $$dF(t,X_t) = F_t(t,X_t)dt + F_x(t,X_t)dX_t + \tfrac{1}{2}F_{xx}(t,X_t)d\sigma_t^2 ,$$

i.e.

$$F(t_2,X_{t_2}) - F(t_1,X_{t_1}) = \int_{t_1}^{t_2} F(t,X_t)dt + \int_{t_1}^{t_2} F_x(t,X_t)dX_t + \tfrac{1}{2}\int_{t_1}^{t_2} F_{xx}(t,X_t)d\sigma_t^2$$

for all $a \le t_1 \le t_2 \le b$, where the first and the third integrals are mean square or sample path integrals. The condition of $F(t,x)$ may be relaxed when a specific R is given.

It may seem surprising that this differential formula does not involve the quadratic variation V of X. However, V is used implicitly in view of (7.4).

Note that if X is a Gaussian martingale then (8.1) coincides with the differential formula for martingales, and that if $\sigma_t^2 = R(t,t)$ is constant then (8.1) is the same differential formula as in the usual calculus.

Finally suppose that X has sample paths of bounded variation and $G(t,x)$ is a continuous function satisfying (7.3). Then it follows from (8.1) that

$$L\int G(t,X_t)dX_t = \int G(t,X_t)dX_t + \tfrac{1}{2}\int G_x(t,X_t)d\sigma_t^2$$

where $L$ indicates Lebesgue integral. If, in addition, $\sigma_t^2$ is constant then the stochastic integral equals the Lebesgue integral.

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| | |
|---|---|
| 1. REPORT NUMBER: 18 AFOSR TR-78-0516 — 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle): 6 GAUSSIAN PROCESSES: NONLINEAR ANALYSIS AND STOCHASTIC CALCULUS. | 5. TYPE OF REPORT & PERIOD COVERED: 9 Interim rept.; |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s): 10 Stamatis Cambanis [redacted] Steel T. Huang | 8. CONTRACT OR GRANT NUMBER(s): 15 AFOSR-75-2796 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of North Carolina, Department of STatistics, Chapel Hill, NC 27514 | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 16 2304/A5 17 A5 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, DC 20332 | 12. REPORT DATE: 11 1977 |
| | 13. NUMBER OF PAGES: 13 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | 15. SECURITY CLASS. (of this report): UNCLASSIFIED 12 15p. |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Gaussian processes, multiple Wiener integrals, nonlinear systems, stochastic integrals, differential formula

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is a survey of recent work on nonlinear analysis and stochastic calculus for Gaussian processes. Topics included are multiple Wiener integrals for Gaussian processes and their use in nonlinear system representation and identification, and stochastic integrals for Gaussian processes and their differential formula.

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